Toward a language theoretic proof of the four color theorem

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outline of the talk

1. introduction
2. parameterized families of trees
3. reducing a pair of trees
the context-free grammar $G$

start symbols: 0, 1, 2
formation rules: $0 \rightarrow 12$, $0 \rightarrow 21$, $1 \rightarrow 02$, $1 \rightarrow 20$, $2 \rightarrow 01$, $2 \rightarrow 10$

An $n$-leaf tree $T$ parses a length-$n$ word $w$ on $\{0, 1, 2\}$ if $T$ is a valid derivation tree for $w$ under $G$.

For example, the tree parses 0110212:

The set of possible derivation trees under $G$ is the set of binary trees.
The grammar $G$ is ambiguous; there exist distinct trees that parse a common word.

The trees

both parse 010:
another example

The trees

both parse 0110212:
Theorem

Let $n \geq 1$, and let $T_1$ and $T_2$ be $n$-leaf binary trees. Then $T_1$ and $T_2$ parse a common word under $G$. 

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Theorem (Louis Kauffman, 1990)

The following are equivalent.

- Every pair of n-leaf binary trees parses a common word under G.
- Every planar map is four-colorable.

Perhaps an enumerative or language theoretic approach will lead to a proof of the four color theorem that is shorter than known proofs.
sketch of correspondence
\{e, 0, 1, 2\} is the Klein 4-group.
sketch of correspondence
Let $\text{ParseWords}(T_1, T_2)$ be the set of equivalence classes of words parsed by both trees $T_1$ and $T_2$.

For example, $\text{ParseWords}(\text{ }, \text{ }) = \{0121\}$.

The four color theorem is equivalent to the statement that for every pair of $n$-leaf binary trees $T_1$ and $T_2$ we have $\text{ParseWords}(T_1, T_2) \neq \{\}$. 
outline of the talk

1. introduction

2. parameterized families of trees

3. reducing a pair of trees
A path tree is a binary tree with at most two vertices on each level.

Let $\text{LeftCombTree}(n)$ be the $n$-leaf path tree corresponding to $l^{n-2}$.

Let $\text{RightCombTree}(n)$ be the $n$-leaf path tree corresponding to $r^{n-2}$. 
Theorem

\[ \text{ParseWords}(\text{LeftCombTree}(n), \text{RightCombTree}(n)) = \begin{cases} \{01^{n-2}2\} & \text{if } n \geq 2 \text{ is even} \\ \{01^{n-2}0\} & \text{if } n \geq 3 \text{ is odd}. \end{cases} \]

Proof by example.
a pair of comb trees

**Theorem**

\[
\text{ParseWords}\left(\text{LeftCombTree}(n), \text{RightCombTree}(n)\right) = \\
\begin{cases} 
\{01^{n-2}2\} & \text{if } n \geq 2 \text{ is even} \\
\{01^{n-2}0\} & \text{if } n \geq 3 \text{ is odd.}
\end{cases}
\]

*Proof by example.*

![Comb Tree Diagram]
Theorem

ParseWords(LeftCombTree(n), RightCombTree(n)) =

\[
\begin{cases}
\{01^{n-2}2\} & \text{if } n \geq 2 \text{ is even} \\
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Proof by example.
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**Proof by example.**
a pair of comb trees

**Theorem**

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\]

*Proof by example.*

```
0 2 1
0 1
```
```
0 2
1 0
1 2
```
Theorem

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\begin{cases} 
\{01^{n-2}2\} & \text{if } n \geq 2 \text{ is even} \\
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Proof by example.
Theorem

\[ \text{ParseWords}(\text{LeftCombTree}(n), \text{RightCombTree}(n)) = \begin{cases} \{01^{n-2}2\} & \text{if } n \geq 2 \text{ is even} \\ \{01^{n-2}0\} & \text{if } n \geq 3 \text{ is odd.} \end{cases} \]

Proof by example.

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crooked trees

Let $\text{LeftCrookedTree}(n)$ be the path tree corresponding to $(lr)^{(n-2)/2}$.

\[ \ldots \]

Let $\text{RightCrookedTree}(n)$ be the path tree corresponding to $(rl)^{(n-2)/2}$.

\[ \ldots \]
a comb tree and a crooked tree

Theorem

\[
\text{ParseWords}(\text{LeftCombTree}(n), \text{RightCrookedTree}(n)) =
\begin{cases}
\mod(1 - n, 3) \left( (012)^{n/6} \right)^R (012)^{(n-2)/6} & \text{if } n \geq 2 \text{ is even} \\
\mod(1 - n, 3) \left( (012)^{(n-3)/6} \right)^R (012)^{(n+1)/6} & \text{if } n \geq 3 \text{ is odd.}
\end{cases}
\]
The number of parse words is generally not constant.

**Theorem**

For $n \geq 2$,

$$|\text{ParseWords}(\text{LeftCrookedTree}(n), \text{RightCrookedTree}(n))| = 2^{\lfloor n/2 \rfloor} - 1.$$
Let $\text{LeftCombTree}(m, n)$ and $\text{RightCombTree}(m, n)$ be the $(m + n)$-leaf path trees corresponding to $l^m r^{n-2}$ and $r^m l^{n-2}$.

For example, $\text{LeftCombTree}(3, 5) = \text{ }$.

**Theorem**

For $m \geq 1$, $k \geq 1$, and $n \geq k + 2$,

$$b(m, k) = |\text{ParseWords(LeftCombTree}(m, n), \text{RightCombTree}(k, m + n - k))|$$

is independent of $n$. Moreover,

$$(M - 2)(M - 1)(M + 1) b(m, k) = 0.$$
1. introduction

2. parameterized families of trees

3. reducing a pair of trees
If two trees have subtrees with the same sets of leaves, we can decompose the pair into smaller pairs.
decomposable pairs

Breaking the trees as

produces the same partition \( \{a, l\}, \{b, c, h, i, j, k\}, \{d, e, f, g\} \) of the leaves in both trees.
extending a parse word . . .

Consider the pair

which parses 01220. “Triplicate” the first leaf:

We have extended the parse word for the smaller pair to a parse word for a larger pair.
Since the (larger) pair contains the right comb in leaves 1–3, it is reducible to

\[ \text{Theorem} \]

*If a pair of n-leaf (not necessarily path) trees has three consecutive leaves that appear in a comb structure in both trees, then the pair is reducible.*

In particular, the three leaves receive the same label for some parse word.
“mutual crookedness”

However, something stronger appears to be true.

Conjecture

If a pair of $n$-leaf trees has two consecutive leaves that appear in a comb structure in both trees, then there is a parse word in which the two leaves receive the same label.

But there is no obvious relationship between the parse word of the original pair and the parse word of the “reduced” pair!
To prove the “four color theorem for path trees” it suffices to consider indecomposable, “weakly mutually crooked” pairs of path trees.

Existing proofs of the four color theorem successfully use the notion of reducibility. Should the language theoretic approach also pursue it?

Since the number of parse words of $\text{LeftCombTree}(m, n)$ and $\text{RightCombTree}(k, m + n - k)$ satisfies such a simple recurrence, looking for generalizations seems like a promising direction.