A Method for Generating Integer Solutions to Matrix Equations

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Goal: to generate all integer matrices X that satisfy the equation $AX = B$ for integer matrices A and B, where A is $m \times n$ and B is $m \times p$.

For any integer matrix A, there are invertible integer matrices P and Q such that

$$PAQ = D,$$

where D is a diagonal matrix of integers, and for which Q^{-1} is an integer matrix.

The matrix D can be made to have the property that

$$D = \text{diag}\{d_{11}, d_{22}, \cdots, d_{rr}, 0, \cdots, 0\}$$

with d_{ii} a factor of d_{jj} for $i < j$ and for which $d_{ii} \neq 0$ for $i \leq r$, but we have no need for the divisibility criterion.

Nathan Jacobson, *Lectures in Abstract Algebra, Vol. 2: Linear Algebra,* Van Nostrand, 1953, p. 79.

Let

$$\overline{D} = \text{diag}\{d_{11}, d_{22}, \cdots, d_{rr}\}$$

and

$$PB = \begin{pmatrix} \overline{PB} \\ U \end{pmatrix},$$

where $\overline{PB}$ is the matrix consisting of the first r rows of PB.

Theorem: $AX = B$ for the integer matrix X if and only if

a) $U = O$, a matrix of zeros,

b) $\overline{D}^{-1}\,\overline{PB}$ is an integer matrix, and

c) $X = Q \begin{pmatrix} \overline{D}^{-1}\,\overline{PB} \\ Z \end{pmatrix}$ for some integer matrix Z.

Operations: Multiply A on the left and right by integer elementary matrices with integer inverses, which is equivalent to

1) swapping rows,

2) swapping columns,

3) changing the sign of a row or column,

4) using row operations of the form $\beta R_j + R_i \to R_i$, and

5) using column operations of the form $\beta C_j + C_i \to C_i$, where β is an integer.

Our modification of Jacobson's algorithm allows division of a row by a constant to make $d_{ii} = 1$ for $i \leq r$. As a result, P may not necessarily be an integer matrix.

Strategy: Starting with the matrix A, we carry through the columns the repeated application of the Euclidean algorithm to the entries in the first row of A.

With each application we retain the remainder when each entry is divided by the entry corresponding to the entry in the first row of A with smallest absolute value.

Eventually this will produce a matrix in which the entries of the first row are all 0, except for one entry, which is the greatest common divisor of the first row of A.

By dividing the first row of A by this element and interchanging columns, we produce a matrix with a 1 in the pivot position and 0 elsewhere in the first row.

We then use row operations to change the other elements in the first column of A to 0.

Corollary: $AX = B$ for the integer matrix X if and only if

a) $U = O$, a matrix of zeros,

b) $\overline{PB}$ is an integer matrix and

c) $X = Q \begin{pmatrix} \overline{PB} \\ Z \end{pmatrix}$ for some integer matrix Z.

Procedure: Form the array

$$\begin{bmatrix} A & B \\ I_n & * \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix, and * is left blank.

Result:

$$\begin{bmatrix} D & PB \\ Q & * \end{bmatrix} = \begin{bmatrix} I_r \; O & \overline{PB} \\ O \; O & U \\ Q & * \end{bmatrix}.$$

where $D = \text{diag}\{1, 1, \cdots, 1, 0, \cdots, 0\}$ is a matrix of rank r.

If U is not a matrix of 0's or if $\overline{PB}$ is not an integer matrix, then there are no integer solutions.

Example: $A = \begin{bmatrix} 6 & 7 & 8 \\ 8 & 17 & 16 \\ 10 & 27 & 24 \end{bmatrix}$ and

$B = \begin{bmatrix} -29 & 50 \\ -53 & 84 \\ -77 & 118 \end{bmatrix}$.

Begin with

$$\left[\begin{array}{ccc|cc} 6 & 7 & 8 & -29 & 50 \\ 8 & 17 & 16 & -53 & 84 \\ 10 & 27 & 24 & -77 & 118 \\ 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \end{array}\right].$$

$$\left[\begin{array}{rrr|rr} 1 & 0 & 0 & -29 & 50 \\ 0 & -46 & -10 & 208 & -366 \\ 0 & -92 & -20 & 416 & -732 \\ -1 & 7 & 1 & & \\ 1 & -6 & -2 & & \\ 0 & 0 & 1 & & \end{array}\right]$$

$$\left[\begin{array}{rrr|rr} 1 & 0 & 0 & -29 & 50 \\ 0 & 1 & 0 & -104 & 183 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 5 & -12 & & \\ 1 & 6 & -16 & & \\ 0 & -9 & 23 & & \end{array}\right]$$

Now $\overline{PB} = \begin{bmatrix} -29 & 50 \\ -104 & 183 \end{bmatrix}$, and

$$X = QY = \begin{bmatrix} -1 & 5 & -12 \\ 1 & 6 & -16 \\ 0 & -9 & 23 \end{bmatrix} \begin{bmatrix} -29 & 50 \\ -104 & 183 \\ z_1 & z_2 \end{bmatrix}$$

$$= \begin{bmatrix} -491 - 12z_1 & 865 - 12z_2 \\ -653 - 16z_1 & 1148 - 16z_2 \\ 936 + 23z_1 & -1647 + 23z_2 \end{bmatrix}$$

where z_1 and z_2 are arbitrary integers.

Since $491/12 \approx 41$, replace z_1 with $z_1 - 41$. Similarly, replace z_2 with $z_2 + 72$.

$$X = \begin{bmatrix} 1 - 12z_1 & 1 - 12z_2 \\ 3 - 16z_1 & -4 - 16z_2 \\ -7 + 23z_1 & 9 + 23z_2 \end{bmatrix}, z_1, z_2 \in N,$$

the complete set of integer solutions.

Gauss-Jordan:

$$\left[\begin{array}{ccc|cc} 6 & 7 & 8 & -29 & 50 \\ 8 & 17 & 16 & -53 & 84 \\ 10 & 27 & 24 & -77 & 118 \end{array}\right]$$

reduces to

$$\left[\begin{array}{ccc|cc} 1 & 0 & 12/23 & -61/23 & 131/23 \\ 0 & 1 & 16/23 & -43/23 & 52/23 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so that

$$X = \begin{bmatrix} (-61 - 12z_1)/23 & (131 - 12z_2)/23 \\ (-43 - 16z_1)/23 & (52 - 16z_2)/23 \\ z_1 & z_2 \end{bmatrix}.$$