Supplement D - The Fifteen Puzzle

Overview

The purpose is not to explain how to solve the Fifteen Puzzle (with a little experience, it's not difficult to solve,) but to determine if a given configuration is solvable.

Procedure

For simplicity, let's start with an "Eight Puzzle" as illustrated below (Figure 1.)

6	2	3
	5	4
1	7	8

Figure 1

We list the numbers from left to right and top to bottom, treating the blank as the number 9. This gives:

6, 2, 3, 9, 5, 4, 1, 7, 8

Beginning with the first number in the sequence, we compute the number of **backward pairs.** 6,2 6,3 6,5 6,4 and 6,1 are backward pairs. So are 2,1 3,1 9,5 9,4 9,1 9,7 9,8 5,4 5,1 and 4,1. In all there are 15 backward pairs. Let BP denote the number of backward pairs. In this example, BP = 15.

Next, treat the squares as a checker board with alternating red and black squares as illustrated in Figure 2. For the Fifteen Puzzle this pattern is given in Figure 3.

R	B	R
B	R	B
R	B	R

Figure 2

R	B	R	B
B	R	B	R
R	B	R	B
B	R	B	R

Figure 3

The puzzle is solvable if one of the two conditions hold:

1. BP is even and the blank is on a red square, or
2. BP is odd and the blank is on a black square.

Otherwise, the puzzle is not solvable. In the above example, BP is odd and the blank is on a black square. Therefore the puzzle is solvable.

Exercises

In exercises 1-6, determine if the given puzzles are solvable:

1.

2	3	1
4	6	7
	8	5

2.

5	6	7
8		3
1	2	4

3.

	8	7
1	2	3
4	5	6

4.

6	2	3
	5	4
1	7	8

5.

4	12	6	3
15	1	2	5
7	8	10	9
11	13		14

6.

1	2	3	4
5	6	7	8
13	14	15	
9	10	11	12

7. Verify the solvability of the fifteen puzzle generated in the computer lab.