11 · 10 · 2003

Current Research Problems

Boundary impurity models and geometric entropy. The geometric (or *entanglement*) entropy of spin chains and related $1+1$ dimensional massless models depend universally upon the central charge of the theory and diverge logarithmically with the length of the subsystem. Specifically, the entropy is given by $S = -\mathrm{tr}\rho\log\rho \sim c\log L/\epsilon$, where the reduced density matrix ρ has been formed by tracing over the degrees of freedom exterior to a subsystem of size L, and ϵ is a UV cut-off. In the work of Kane and Fisher (1993), a single impurity in a Luttinger Liquid (LL) was shown to have a dramatic effect—either effectively decoupling the two sides, or effectively vanishing, depending on whether the LL is repulsive or attractive. How does the geometric entropy behave if an impurity is introduced at the boundary of the subsystem?

If one considers a "mixed" real space/momentum space basis where at wavenumber k there are a series of (kL) spatial modes of width k^{-1}, entanglement entropy arises from those spatial modes which "straddle" the boundary; counting them naively gives the logarithmic behavior stated above. The UV divergence of the entropy is a consequence of the unlimited number modes close the boundary. (An identical divergence arises in computing the entanglement entropy across the event horizon of a black hole.) If an impurity is introduced into a repulsive LL, the effective strength grows with inverse wavenumber and therefore the contributions to entanglement become weak at the longest wavelengths. Such an effect should weaken the logarithmic behavior above. Analyzing this problem perturbatively (within RG) is problematic due to the non-local nature of the entropy; however a numerical approach may be possible based upon a euclidean version of this problem.

Callan and Wilczek (1993) introduced an elegant euclidean path integral representation of the geometric entropy for a $1+1$ dimensional field theory restricted to the semi-infinite spatial domain. The reduced density matrix, $\langle\phi_+|\rho|\phi_-\rangle$ may be expressed as path integral over the *entire* $1+1$ dimensional euclidean space $(x, \tau \in (-\infty, \infty))$ with $\phi_\pm$ as boundary conditions on either side of a cut along the polar angle $\theta = 0$. If this cut is enlarged to a finite "deficit angle," δ, the path integral is now evaluated on a cone. Using the replica trick they showed that the entropy $S = -\mathrm{tr}\rho\log\rho$ may be written as a variation of the partition function, Z_δ, with respect to δ. Specifically, $S = (2\pi\frac{d}{d\delta} + 1)\log Z_\delta|_{\delta=0}$. Remarkably, this expression has the same form as thermodynamic entropy, $S = (-\beta\frac{\partial}{\partial\beta} + 1)\log Z$, where β is identified with the complete polar angle 2π. The entropy in the presence of an impurity added at the origin may then be evaluated by euclidean Monte-Carlo sampling of the bosonic LL action, but on a surface with a conical singularity at the origin.

Based upon the work of Kabat and Strassler (1994), t'Hooft (1984) and more recently Gaite (2003), there may *also* be a way of analytically studying the action proposed above. If one chooses to represent the action in the full euclidean space by polar coordinates, a Hamiltonian that generates translations in θ may be found by identifying $\partial_\theta\phi$ with the canonical momentum. (This Hamitonian generates *time* translations of a sort—in Minkowski space, the boost angle θ is the proper time of a uniformly accelerated observer). The Hamiltonian H_r for a particular time slice is now the radial section of the action, and, since time is periodic, the reduced density matrix is at temperature $1/2\pi$: $\langle\phi_+|\rho|\phi_-\rangle = \langle\phi_+|e^{-2\pi H_r}|\phi_-\rangle$. The problem of evaluating the entropy in the presence of an impurity becomes that of the finite temperature thermodynamic entropy of a gas described by the radial Hamiltonian H_r, with an impurity at the origin.

Qubit-Oscillator Model and the Fluctuating Gap Model. Models of dissipative quantum environments such as the spin-boson model (SBM) are intentionally weakly coupled ($O(N^{-1/2})$) and therefore neglect the back-action of the qubit on the environment. However, environments imposed by the quantum detectors in qubit gates may be strongly coupled and exhibit sharp spectral features. In cond-mat/0301463 (with V. N. Muthukumar) we studied the problem of how a qubit becomes entangled with a single oscillator mode when the qubit back-action on the environment is not neglected. The resulting model seems to be closely related to the Fluctuating Gap Model (FGM)—a model of noninteracting fermions where the superconducting (or Peierls) gap exhibits spatial disorder.

If the coupling is strong enough, the qubit can force a symmetry change upon the potential of a sufficiently adiabatic oscillator and an instanton emerges in the collective qubit-oscillator degree of freedom. This is just a simple form of entanglement. The fluctuations of the environment have now become non-Gaussian and, unlike weakly coupled models like the SBM, the correlation functions of the qubit cannot be found by integrating out the the "bare" environment degrees of freedom. To compute the correlation function for the qubit, we must first solve for the correlation function in the presence of an instanton in the oscillator coordinate and then average over all instanton configurations. The imaginary time qubit-oscillator action for the simple model we considered may be written,

$$S = \int d\tau \bar{\psi}(\tau)(\partial_\tau + t\sigma_x + \lambda x(\tau)\sigma_z)\psi(\tau) + S_0$$

where $\psi(\tau)$ are spinors representing the qubit, $x(\tau)$ is the oscillator displacement and S_0 is the bare oscillator action. Integrating out the qubit produces a bistable oscillator effective potential and a tunneling solution is stabilized for large mass and dimensionless coupling constant. The correlator for the oscillator becomes $\langle T_\tau x(\tau)x(\tau')\rangle_{S_{\mathrm{eff}}} = x_0^2 e^{-\Gamma|\tau-\tau'|}$, where Γ is the tunneling frequency.

The equation of motion for the fermion Green's function in the Fluctuating Gap Model is

$$[i\partial_x + \omega\sigma_x + i\Delta(x)\sigma_z]G(x, x', \omega) = \delta(x - x')$$

where the gap function $\Delta(x)$ has a disorder covariance $\langle\Delta(x)\Delta(x')\rangle = \Delta_0^2 e^{-|x-x'|/\xi}$ and ξ is the disorder correlation length. The relationship to the FGM may now be seen: Replacing the 1-d spatial degree of freedom in the FGM by imaginary time, the qubit action is the action for a fermion of frequency $i\Delta$ propagating in a Peierls chain with a gap function $\lambda x(\tau)$. Averaging the qubit correlation function—computed with a "frozen" oscillator configuration—over the fluctuations of $x(\tau)$ with the correlator above is equivalent to the disorder average with a correlation length given by the instanton time, Γ^{-1}. In the FGM, the density of states at the Fermi surface is accessed by the zero frequency limit $\omega \to i0^+$ which corresponds to $t \to 0^+$, the strong coupling limit in the qubit problem.

The object of interest in the FGM is the behavior of the density of states at the Fermi surface; for finite range disorder and zero DC gap (the condition most relevant to the mapping), symmetry dictates a divergence in the density of states. However, the two models differ in that the correlation length of the gap disorder in the FGM is an independently controlled parameter whereas the instanton fluctuations are generated spontaneously in the qubit model. In addition, the fermions in the qubit model satisfy twisted boundary conditions (this the Fedotov-Popov representation of spin-$\frac{1}{2}$). Nonetheless, this relationship may provide a novel way of studying the density of states singularity in the FGM by a zero-dimensional quantum mechanics problem.